# COT 3100C Homework #9

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# 1 Question 1

Johnny has rolled a pair of standard fair six-sided dice. Given that one of the two dice shows 3 dots, what is the probability that the sum of the number of dots showing on both dice is even?

Intuition would say $\frac{1}{2}$. However, let's make a lattice diagram:

|   | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 |

We see that of the 36 possible combinations of rolling 2 die, 11 of those have at least 1 dice rolled as a 3. Of the 11, 5 of those have a sum that is even (4, 4, 6, 8, 8). Thus the probability that the sum of the number of dots showing on both dice is even is:

$$= \frac{5}{11}$$

# 2 Question 2

In class, we played a game where 3 distinct digits were hidden behind 3 doors. The contestant would choose one of the doors and the number behind it was revealed. Based on this number, the contestant could either choose to stay with that door, or switch to another door, with the goal of selecting the door that had the lowest number behind it. We also showed in class that the optimal strategy for the game was to keep the same door if the number revealed was 3 or lower, and to switch otherwise. Change the game so that the numbers behind the 3 doors are distinct positive integers in between 1 and 30, inclusive. What is the largest integer k, such that if a contestant reveals k, their best chance of winning is staying with that door. For that value of k, what is the contestants chance of winning, given that they stay with the same door? For that value of k, what is the contestants chance of winning if they switch to a different door?

Let's say person reveals k.

$$P(Win \mid Stay) = \frac{\binom{30-k}{2}}{\binom{29}{2}}$$

$$= \frac{\frac{(30-k)!}{2!(28-k)!}}{406}$$

$$= \frac{(29-k)(30-k)}{812}$$

Now if you stay, there are two doors that could reveal a higher number than k:

$$P(Win \mid Switch) = \frac{1 - P(Win \mid Stay)}{2}$$

$$= \frac{1 - \frac{(29-k)(30-k)}{812}}{2}$$

$$= \frac{1}{2} - \frac{(29-k)(30-k)}{1624}$$

We want $P(Win \mid Stay) \geq P(Win \mid Switch)$:

$$\frac{(29-k)(30-k)}{812} \geq \frac{1}{2} - \frac{(29-k)(30-k)}{1624}$$

$$2(29-k)(30-k) \geq 812 - (29-k)(30-k)$$

$$3(29-k)(30-k) \geq 812$$

$$k^2 - 30k - 29k + 870 - \frac{812}{3} \geq 0$$

$$k^2 - 59k + \frac{1798}{3} \geq 0$$

$$k = \left\lceil \frac{59 \pm \sqrt{59^2 - 4(\frac{1798}{3})}}{2} \right\rceil$$

$$= 13, 45$$

Thus, the equality is true when $k \leq 13$ as 45 is not a possible number.

Now plug back 13 to k:

Probability of winning if we stay with the same door:

$$= P(Stay \mid Win) = \frac{(29-13)(30-13)}{812} \approx 0.334975369458$$

Probability of winning if we switch to another door:

$$= P(Switch \mid Win) = \frac{1}{2} - \frac{(29-13)(30-13)}{1624} \approx 0.332512315271$$

# 3   Question 3

Consider the following discrete random variable X:
X = 2, with probability 0.1
= 8, with probability 0.2
= 12, with probability 0.5
= 16, with probability 0.2
Calculate both E(X) and Var(X):

$$E(X) = 2(0.1) + 8(.2) + 12(0.5) + 16(0.2) = 11$$

$$E(X^2) = 4(0.1) + 64(0.2) + 144(0.5) + 256(0.2) = 136.4$$

$$Var(X) = E(X^2) - (E(X))^2 = 136.4 - 11^2 = 15.4$$

# 4   Question 4

Sam's probability of getting an A on an individual test is 75%. If he takes 12 tests, what is the probability he gets As on exactly 10 of those tests?

There is a binomial distribution, where there is a fixed and independent probability of success (p = .75) and observations (n = 12).

Thus, the probability he gets As in exactly k = 10 of those tests is:

$$= P(X = k) = \binom{12}{10}(.75)^{10}(1 - .75)^{(12-10)} \approx 0.232293248177$$

# 5   Question 5

Here are the rules for Arups Game of Dice:

1) Roll a pair of dice.
2) If you roll a sum of 11 or 12, you win.
3) If you roll a sum of 2, you lose.
4) Otherwise, record what you've rolled. Let this sum be k; also known as your point.
5) Roll one more time. If this roll exceeds your point(k), you win!
6) If this roll is the same as your point(k) or lower, you lose.
Calculate the probability that a player wins Arups Game of Dice.

Let's refer back to our lattice diagram:

|   | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 |

Now we can make a table for the probabilities for each sum(p) and the probabilities for beating each sum in the second roll(q). (For each col in table, count how many in lattice diagram has the exact sum, and how many in lattice diagram has strictly greater sum):

| 1st roll sum | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| p | 1/36 | 2/36 | 3/36 | 4/36 | 5/36 | 6/36 | 5/36 | 4/36 | 3/36 | 2/36 | 1/36 |
| q | 35/36 | 33/36 | 30/36 | 26/36 | 21/36 | 15/36 | 10/36 | 6/36 | 3/36 | 1/36 | 0/36 |

First, we find the probability of winning first try:

$$= \frac{2}{36} + \frac{1}{36} = \frac{3}{36}$$

Now, we find the probability of winning second try:

$$= 0 + \frac{2}{36} \cdot \frac{33}{36} + \frac{3}{36} \cdot \frac{30}{36} + \frac{4}{36} \cdot \frac{26}{36} + \frac{5}{36} \cdot \frac{21}{36} + \frac{6}{36} \cdot \frac{15}{36} + \frac{5}{36} \cdot \frac{10}{36} + \frac{4}{36} \cdot \frac{6}{36} + \frac{3}{36} \cdot \frac{3}{36} + 0 + 0 + 0$$

$$= \frac{538}{1296}$$

Now, add up the probabilties to get the final answer:

$$\frac{3}{36} + \frac{538}{1296} = \frac{323}{648} \approx 0.498456790123$$

# 6 Question 6

Define a continuous random variable as follows:

$$f(x) = \frac{1}{9}x^2, \text{ for } 0 \le x \le 3, \text{ and}$$

$$= 0, \text{otherwise}$$

Determine both the expectation (E(X)) and variance (Var(X)) of the continuous random variable defined above.

The definition of the expected value is given by:

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

Since the given continuous random variable is bounded from $[0, 3]$, the expected value is:

$$E(X) = \int_0^3 xf(x)dx = \int_0^3 x\frac{1}{9}x^2dx = \int_0^3 \frac{1}{9}x^3dx$$

$$= \frac{1}{36}x^4\Big|_0^3 = \frac{3^4}{36} = \frac{9}{4}$$

The definition of the variable is given by:

$$Var(X) = \int_{-\infty}^{\infty}(x - E(X))^2f(x)dx$$

Since the given continuous random variable is bounded from $[0, 3]$, the variance is:

$$Var(X) = \int_0^3 (x - E(X))^2 f(x)dx = \int_0^3 (x - \frac{9}{4})^2\frac{1}{9}x^2dx$$

$$= \frac{1}{9}\int_0^3 (x^2 - \frac{9}{4}x)^2dx$$

$$= \frac{1}{9}\int_0^3 (x^4 - \frac{9}{2}x^3 + \frac{81}{16}x^2)dx$$

$$= \frac{1}{9}\left(\frac{x^5}{5} - \frac{9x^4}{8} + \frac{81x^3}{48}\right)\Big|_0^3$$

$$= \frac{1}{9}\left(\frac{3^5}{5} - \frac{9(3)^4}{8} + \frac{81(3)^3}{48}\right) = \frac{27}{80} = 0.3375$$

# 7 Question 7

Suppose E and F are events in a sample space and p(E) = 1/3, p(F) = 2/5, and p(F | E) = 1/2. Find p(E | F):

Using Bayes' Theorem:

$$p(E \mid F) = \frac{p(F \mid E) \cdot p(E)}{p(F)} = \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{2}{5}} = \frac{5}{12} \approx 0.416666666667$$

# 8 Question 8

Give a summary of the life (thus far!) and mathematical contributions of Jennifer Balakrishnan:

Jennifer Balakrishnan is an American mathematician born in Mangilao, Guam. Her father is a chemist at the University of Guam. She was gifted in mathematics at a young age, creating a project fpr elliptic coordinate systems as a high school junior and winning the National High School Student Calculus Competition as a high school senior. She went to Harvard University and graduated with both a bachelor's and master's degree in mathmematics in 2006. She then got her PhD in mathematics in 2011 at the Massachusetts Institute of Technology, with a focus on integration for hyperelliptic curves. She become an Assistant Professor at the Boston University in 2016 and promoted to a Associate Professor in 2021.

Balakrishnan has done a vast amount of contribution to the field of mathematics, partictually in computational number theory and artihmetic geometry. One of her better known works is leading a team that solved the "cursed curve", a natural and significant equation in the number theory of elliptic curves. She is one of the mathematicans in the Simons Collaboration on Arithmetic Geometry, Number Theory, and Computation. Her awards include Fellow of the American Mathematical Society (2022), AWMMicrosoft Research Prize in Algebra and Number Theory (2022), and Fellow of the Association for Women in Mathematics (2023), mainly for her work in arithmetic geometry and number theory.

Sources:
1. https://en.wikipedia.org/wiki/Jennifer_Balakrishnan
2. http://math.bu.edu/people/jbala/